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Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry

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Abstract

Suppose that $2d-2$ tangent lines to the rational normal curve $z \mapsto (1 : z : \ldots : z^d)$ in d-dimensional complex projective space are given. It was known that the number of codimension 2 subspaces intersecting all these lines is always finite; for a generic configuration it is equal to the d^{th} Catalan number. We prove that for real tangent lines, all these codimension 2 subspaces are also real, thus confirming a special case of a general conjecture of B. and M. Shapiro. This is equivalent to the following result:

If all critical points of a rational function lie on a circle in the Riemann sphere (for example, on the real line), then the function maps this circle into a circle.

1. Introduction

Two rational functions f_1 and f_2 will be called equivalent if $f_1 = \ell \circ f_2$, where ℓ is a fractional-linear transformation. Equivalent rational functions have the same critical sets.

Theorem 1. *If all critical points of a rational function f are real, then f is equivalent to a real rational function.*

Lisa Goldberg [11] addressed the following question: how many equivalence classes of rational functions of degree d with a given critical set of $2d-2$ points may exist? She reduced this to the following problem of enumerative geometry:

Problem P. *Given $2d-2$ lines in general position in projective space $\mathbb{CP}^d$, how many projective subspaces of codimension 2 intersect all of them?*

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To explain this reduction, to each rational function

$$f(z)=\frac{a_0+\ldots+a_dz^d}{b_0+\ldots+b_dz^d}$$

of degree d, we associate a projective subspace $H_f\subset\mathbb{C}P^d$ defined by the following system of two equations in homogeneous coordinates

$$\begin{aligned}a_0x_0+\ldots+a_dx_d&=0,\\ b_0x_0+\ldots+b_dx_d&=0.\end{aligned}$$

This gives a bijective correspondence between the set of equivalence classes of rational functions of degree d and the set of subspaces of $\mathbb{C}P^d$ of codimension 2 which do not intersect the image of the *rational normal curve* $E(z)=(1:z:\ldots:z^d)\in\mathbb{C}P^d,\ z\in\mathbb{C}P^1$. By straightforward computation one can verify that z_0 is a critical point of f if and only if the tangent line at $E(z_0)$ to the rational normal curve intersects the subspace H_f.

The answer to Problem P, going back to Schubert [17] (see also [14, 15]), is

$$u_d=\frac{1}{d}\left(\begin{array}{c}2d-2\\ d-1\end{array}\right),\quad \text{the } d^{\text{th}} \text{ Catalan number.} \tag{1}$$

Thus the result is

THEOREM A (Goldberg [11]). *The number of equivalence classes of rational functions of degree d with given $2d-2$ critical points is at most u_d.*

We prove

THEOREM 2. *For given $2d-2$ distinct real points, there exist at least u_d classes of real rational functions of degree d with these critical points.*

Theorems A and 2 imply Theorem 1.

In general, even if the lines in Problem P are real, the subspaces of codimension 2 might not be real [14]. Fulton asked the following general question (see [8, p. 55]): how many solutions of real equations can be real, particularly for enumerative problems? We refer to a recent survey [21] of results related to this question. A specific conjecture for the Problem P was made by Boris and Michael Shapiro (see, for example, [20]): if the lines in question are tangent to the rational normal curve at real points, then all u_d solutions of the problem are real. Our Theorem 2 implies that this conjecture is true.

To reformulate Theorem 1, we write a rational function as a ratio of polynomials without a common factor, $f=f_1/f_0$, and suppose for simplicity that ∞ is not a critical point of f. Then critical points of f coincide with zeros of the Wronski determinant $W(f_0,f_1)=f_0f_1'-f_0'f_1$, and Theorem 1 is equivalent to the following: *if the Wronskian of two polynomials has only real zeros, then*

these polynomials can be made real by a linear transformation with constant coefficients. A more general conjecture of B. and M. Shapiro states that this is true for any number of polynomials. It is not enough to require that the Wronskian has real coefficients. Indeed, if $f_1(z) = z^3 + 3iz^2$ and $f_0(z) = z - i$, then $W(f_0, f_1) = 2z^3 + 6z$ has real coefficients, but no nontrivial linear combination of f_0 and f_1 is a real polynomial.

A general discussion of the B. and M. Shapiro conjectures, with experimental evidence and bibliography, is contained in [19], [20]. For the related problem of pole assignment in the theory of automatic control we refer to [6], [7].

As a corollary from his main result in [18], Sottile proved that there exists an open (in the usual topology) set $X \subset \mathbb{R}^{2d-2}$, such that for $x \in X$ there exist u_d classes of real rational functions of degree d, whose critical set is given by x. Theorem 2 was proved by Sottile for $d = 3$, and tested, using computers, for $d \leq 9$. The computation for $d = 9$ ($u_9 = 1,430$) is due to Verschelde [23].

It is interesting that our proof of Theorem 1 is based on the fact that two different enumerative problems have the same sequence of integers as their solution. These two problems are Problem P and the one in Lemma 1 below. We prove Theorem 2 in Sections 2–6 and derive Theorem 1 in Section 7.

The scheme of our proof of Theorem 2 is following. We consider the unit circle $\mathbb{T}$ instead of the real line. Let R be the set of rational functions of degree d, mapping $\mathbb{T}$ into itself, having $2d-2$ distinct critical points in $\mathbb{T}$, and being properly normalized. For $f \in R$ we introduce a "net" $\gamma(f) = f^{-1}(\mathbb{T})$, considered modulo symmetric (with respect to $\mathbb{T}$) normalized homeomorphisms of the Riemann sphere, preserving orientation. A net partitions the Riemann sphere into simply-connected regions; each of these regions is mapped by f homeomorphically onto a component of $\overline{\mathbb{C}}\backslash\mathbb{T}$. Equivalence classes of nets are combinatorial objects, describing topological properties of rational functions $f \in R$. To describe a function $f \in R$ completely, we need one more piece of data, which we call a labeling. It is a function on the set of edges of a net, which assigns to each edge the length of its image. We give a precise description of all nets γ (modulo equivalence) and labelings which may occur. It is important that, for a fixed γ, the space of possible labelings has simple topological structure: it is a convex polytope. To recover f from its labeled net, we first construct a ramified covering $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which maps each edge of γ homeomorphically onto an arc of the unit circle, whose length is specified by the label of this edge. Furthermore g maps each component of the complement $\overline{\mathbb{C}}\backslash\gamma$ homeomorphically onto an appropriate component of $\overline{\mathbb{C}}\backslash\mathbb{T}$. Once such ramified covering g is constructed, the Uniformization theorem implies the existence of a homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, such that $f = g \circ \phi^{-1}$.

This construction leads to a parametrization of the set R by equivalence classes of labeled nets. Similar parametrizations for polynomials and trigono-

metric polynomials were studied by Arnold in [2], [3], and for meromorphic functions on arbitrary Riemann surfaces by Vinberg [24], who used the nets. The dual graph of a net of a meromorphic function is known in classical function theory as a "line complex," or a Speiser graph [10], [25]. It is essentially our tree S, which will be described in Section 2.

Nonequivalent nets correspond to nonequivalent rational functions. For a fixed net γ, each labeling defines a rational function of the class R. Taking the critical set of this rational function, we obtain a map Φ from the space of labelings of γ to the space of critical sets on the unit circle. We prove that Φ is surjective. So for a given critical set, each γ gives a rational function of our class R, and it remains to count all possible classes of nets γ. It turns out that there are exactly u_d of them (Lemma 1).

The main difficulty is the proof of surjectivity of Φ. This is achieved by a version of the "continuity method" going back to Poincaré and Koebe (see, for example, [12, Ch. V, §6]), but we have to use different tools from topology. We extend Φ to a map between closed polytopes and show that the extended map is continuous (Sections 3 and 4). This is done using a normal families argument, Lemma 4. An analysis of the boundary behavior of Φ in Section 5 permits us to prove surjectivity using a topological argument in Section 6.

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We prove Theorem 1 only for $d \geq 3$, because it is trivial for $d = 2$, and because our proof would require a modification in this case.

We fix an integer $d \geq 3$. The map $\mathbf{s} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $\mathbf{s}(z) = 1/\bar{z}$ will be called the *symmetry*. A map will be called *symmetric* if it commutes with the symmetry. A set will be called *symmetric* if the symmetry leaves it invariant. All homeomorphisms and ramified coverings of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C}P^1$, except the involutions like $\mathbf{s}$, are assumed to preserve orientation. For a region D we denote by ∂D its oriented boundary (so that the region is on the left). The unit circle $\mathbb{T}$ is always oriented anticlockwise, so $\mathbb{T} = \partial\mathbb{U}$, where $\mathbb{U}$ is the unit disc. The words "distance" and "diameter" refer to the spherical Riemannian metric on the Riemann sphere. It is obtained from the standard embedding of $\overline{\mathbb{C}}$ as the unit sphere in $\mathbb{R}^3$.

2. Nets, labelings and critical sequences

A *cellular decomposition* of a set $X \subset \overline{\mathbb{C}}$ is a finite partition of X into sets, called cells, each of them homeomorphic to an open unit disc $\mathbb{U}^k \subset \mathbb{R}^k$, $k = 0, 1, 2$; (by definition, $\mathbb{U}^0 = \{\text{one point}\}$), and has closure homeomorphic to the closed disc $\overline{\mathbb{U}}^k$. The cells are called vertices, edges and faces, according to their

dimension. The *degree* of a vertex is the number of edges to whose boundaries this vertex belongs. A *net* $\gamma \subset \overline{\mathbb{C}}$ is the union of edges and vertices of some cellular decomposition of $\overline{\mathbb{C}}$, which satisfies conditions N1–N5 below.

N1. γ is symmetric, that is $\mathbf{s}(\gamma) = \gamma$.

N2. $\mathbb{T} \subset \gamma$.

N3. There are $2d-2$ vertices, all belong to $\mathbb{T}$ and have degree 4.

N4. The point $1 \in \mathbb{T}$ is a vertex.

A cellular decomposition which satisfies N1–N4 is completely determined by its net γ, so we permit ourselves to speak of vertices, edges and faces *of a net*. Because of N3, each face G has an even number of boundary vertices. For every γ satisfying N1–N4 we choose certain distinguished elements as follows. Let $v_0 = 1$, and v_1 be the next vertex anticlockwise on $\mathbb{T}$. There is a unique face G_0 in the unit disc, whose boundary contains at least four vertices, v_0 and v_1 among them. Let v_{-1} be the vertex preceding v_0 on ∂G_0. So when tracing ∂G_0 according to its orientation, we consecutively encounter v_{-1}, v_0, v_1 in this order. We also introduce two edges on the boundary of G_0: $e_1 = [v_0, v_1]$ and $e_{-1} = [v_{-1}, v_0]$. One of these two edges, e' belongs to $\mathbb{T}$, the other, e'' does not. Thus we have double notation for these two edges. For every γ satisfying N1–N4 there is a unique choice of the *distinguished elements* $G_0, v_{-1}, v_0, v_1, e_{-1}, e_1, e'$, and e'' (see Figure 1). The vertices of γ will be enumerated as $v_1, \ldots, v_{2d-2}$, anticlockwise on $\mathbb{T}$, so that $v_{2d-2} = v_0$, and $v_{-1} = v_N$, for some $N = N(\gamma) \in [3, 2d-3]$. Our last assumption about nets is the normalization condition

N5. $v_{-1} = e^{-2\pi i/3}, \quad v_0 = 1, \quad$ and $\quad v_1 = e^{2\pi i/3}, \quad$ the cubic roots of 1.

(The particular choice of these three points on $\mathbb{T}$ is irrelevant). Two nets γ_1 and γ_2 are called *equivalent* if there exists a symmetric homeomorphism h of the sphere $\overline{\mathbb{C}}$, such that $h(\gamma_1) = \gamma_2$, and h leaves each cubic root of 1 fixed. Such h induces a bijective correspondence between the cells of the corresponding cellular decompositions, so we can speak of a vertex, an edge or a face of a class of nets. Each distinguished element described above is mapped by h onto a distinguished element with the same name. We denote by $[\gamma]$ the equivalence class of a net γ.

For a net γ we denote by V, E and Q the sets of its vertices, edges and faces, respectively. Euler's formula implies $|Q| = 2d$ and $|E| = 4d-4$. We denote by $Q_{\mathbb{U}} \subset Q$ the subset of faces which belong to $\mathbb{U}$, and by $E_{\mathbb{T}}$ the subset of edges, which belong to $\mathbb{T}$.

Figure 1 shows all nets for $d = 4$ with distinguished faces and vertices. For aesthetic reasons we ignored N5 in this picture.

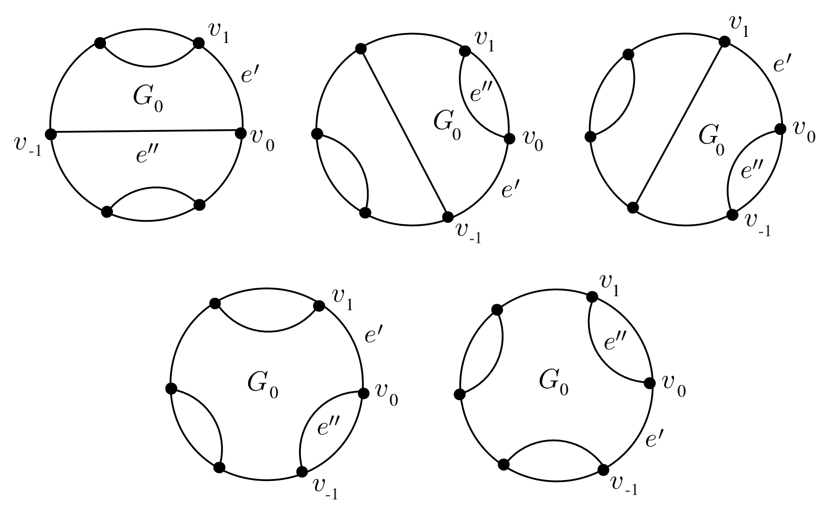

Figure 1. All nets for $d = 4$. Only the parts in $\overline{\mathbb{U}}$ are shown.

LEMMA 1. *There exist exactly u_d classes of nets with $2d-2$ vertices, where u_d is the Catalan number* (1).

This can be found in [22, Exercise 6.19 **n**]. This exercise contains 66 combinatorial problems with the Catalan numbers as the answer (see also Exersise 6.25 for algebraic interpretations of these numbers). Stanley uses notation $C_n = u_{n+1}$.

To each net γ corresponds the *dual graph* S of the cellular decomposition of $\overline{\mathbb{U}}$ defined by γ. More precisely, each vertex $q = q_G$ of S corresponds to a face $G = G_q \in Q_{\mathbb{U}}$, and two vertices of S are connected by an edge $\tau = \tau_e$ in S if the two corresponding faces in $Q_{\mathbb{U}}$ have a common edge $e = e_\tau$ in $\gamma \cap \mathbb{U}$.

Let $\hat{S}$ be the graph obtained by the following extension of S: for every edge $e \in E_{\mathbb{T}}$, a vertex q_e and an edge τ_e connecting q_e with q_G are added to S, where G is the face in $Q_{\mathbb{U}}$ with $e \in \partial G$.

It is easy to see that S and $\hat{S}$ are trees. We designate $q_0 = q_{G_0}$ to be the root of these trees. Notice that the edges of $\hat{S}$ are in bijective correspondence with edges of γ in $\overline{\mathbb{U}}$, and the edges of S correspond to the edges of γ in $\mathbb{U}$. There is a natural partial order on the vertices of a rooted tree, so that the root is the unique minimal element. Thus the tree S defines a partial order on faces in $Q_{\mathbb{U}}$:

$$G' < G \quad \text{if the path in } S \text{ from } q_0 \text{ to } q_G \text{ passes through } q_{G'}. \tag{2}$$

We can also order the set of faces in $Q_{\mathbb{U}}$ into a sequence $G_0, \dots, G_{d-1}$ so that, for every $k \in [1, d-1]$, the face G_k has exactly one common boundary edge with the union of the faces $G_0, \dots, G_{k-1}$. Such ordering is always compatible with the partial order (2):

$$\text{(3)} \qquad \text{for every} \quad m, n \in [0, d-1] \quad G_n < G_m \quad \text{implies} \quad n < m.$$

We will use repeatedly the possibility of such ordering.

For a net we define a function $\sigma : Q \to \{1, -1\}$, called *parity*. We put $\sigma(G_0) = 1$, for the distinguished face, and then $\sigma(G)\sigma(G') = -1$ if the faces G and G' have a common edge on their boundaries. Such parity function exists for every cellular decomposition whose vertices have even degree. With our normalization $\sigma(G_0) = 1$, the parity function is unique.

A *labeling* of a net is a nonnegative function on the set of edges, $p : E \to \mathbb{R}$, satisfying the following conditions:

$$\text{(4)} \qquad p(\mathbf{s}(e)) = p(e) \quad \text{for every} \quad e \in E,$$

where $\mathbf{s}$ is the symmetry with respect to $\mathbb{T}$,

$$\text{(5)} \qquad \sum_{e \subset \partial G} p(e) = 2\pi \quad \text{for every} \quad G \in Q_{\mathbb{U}},$$

and

$$\text{(6)} \qquad p(e_1) = p(e_{-1}) = 2\pi/3.$$

A pair (γ, p) is called a *labeled net*. Two labeled nets (γ_1, p_1) and (γ_2, p_2) are *equivalent* if there exists a symmetric homeomorphism $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, fixing the three cubic roots of 1, and having the properties $h(\gamma_1) = \gamma_2$, and $p_2(h(e)) = p_1(e)$ for every edge e of γ_1.

A labeling p is called *degenerate* if $p(e) = 0$ for some edges $e \in E$, otherwise it is called *nondegenerate*. The space of all labelings is a closed convex polytope $\overline{L}_\gamma$ in the affine subspace A of $\mathbb{R}^{4d-4}$ defined by (4), (5) and (6). Its interior L_γ with respect to A, the set of nondegenerate labelings, is homeomorphic to a cell of dimension $2d - 5$.

The statement about dimension will not be used, but it can be verified in the following way. First, using the equations (4), we eliminate the variables $p(e)$ for all edges in $\overline{\mathbb{C}} \backslash \overline{\mathbb{U}}$. The number of remaining variables is $3d - 3$. Each of the equations (5) corresponds to a face $G \in Q_{\mathbb{U}}$. This face G has at least one boundary edge on $\mathbb{T}$, which does not belong to the boundaries of other faces in $Q_{\mathbb{U}}$. Thus each equation in (5) contains a variable which does not show in other equations. So the codimension of the affine subspace defined by all equations (5) is d. Equations (6) increase the codimension to $d + 2$. So the dimension of L_γ is $2d - 5$.

A *critical sequence* corresponding to γ is a map $c : V \to \mathbb{T}$, which leaves v_0, v_1 and $v_N = v_{-1}$ fixed, and preserves the (nonstrict) cyclic order. We describe critical sequences by nonnegative functions $l : E_{\mathbb{T}} \to \mathbb{R}$, For $k = 1, \dots, 2d-2$, the value $l([v_{k-1}, v_k])$ is defined as the length of the arc $[c(v_{k-1}), c(v_k)]$, of $\mathbb{T}$, described anticlockwise from $c(v_{k-1})$ to $c(v_k)$. This function l has the following properties:

$$(7) \qquad l([v_0, v_1]) = \sum_{k=2}^{N} l([v_{k-1}, v_k]) = \sum_{k=N+1}^{2d-2} l([v_{k-1}, v_k]) = \frac{2\pi}{3}, \quad \text{and} \quad l \geq 0,$$

where $N = N(\gamma)$. Thus we identify the set of all critical sequences with the convex polytope $\overline{\Sigma}_\gamma$, described by (7). This polytope is a product of two simplexes of dimensions $N-2$ and $2d-N-3$, so its dimension is $2d-5$. The interior Σ_γ of our polytope consists of critical sequences with $l > 0$. We call such critical sequences *nondegenerate*, and the critical sequences in $\overline{\Sigma}_\gamma \backslash \Sigma_\gamma$ *degenerate*.

We denote by R^* the class of all rational functions of degree at most d, which preserve the unit circle, whose critical points all belong to the unit circle, and which satisfy the normalization condition

$$(8) \qquad f(z) = z, \quad f'(z) = 0, \quad \text{for} \quad z \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}.$$

This normalization implies that two different functions of the class R^* are never equivalent.

For each class of nets $[\gamma]$, we consider a subclass $R_\gamma \subset R^*$ defined by the following condition:

$$(9) \qquad f^{-1}(\mathbb{T}) \in [\gamma].$$

It follows from (9) that R_γ consists of rational functions of degree d with simple critical points, which coincide with the vertices of $f^{-1}(\mathbb{T})$. Furthermore, (8) and (9) imply that f maps the distinguished face G_0 of the net $f^{-1}(\mathbb{T})$ onto the unit disc.

It will follow from the results of Section 3 that $R_\gamma \neq \emptyset$ for every γ.

3. Construction of a map

$$(10) \qquad F_\gamma : \overline{L}_\gamma \to R^* \times \overline{\Sigma}_\gamma.$$

In this section, for each net γ, we construct a map (10), where $\overline{L}_\gamma, R^*$ and $\overline{\Sigma}_\gamma$ were introduced in Section 2, with the following properties:

$$(11) \qquad F_\gamma(L_\gamma) \subset R_\gamma \times \Sigma_\gamma.$$

If p is a nondegenerate labeling, and $(f, c) = F_\gamma(p)$, then c is the sequence of

critical points of f. An additional property, related to the boundary behavior of F_γ, is stated in Proposition 1 below. In Section 4 we will prove that the second component Φ_γ of F_γ is continuous, and in Section 6 that Φ_γ is surjective.

To construct our map F_γ, we fix γ and a labeling $p \in \overline{L}_\gamma$. We introduce the following notation. Let Z be the union of edges e with $p(e) = 0$, and D the component of $\overline{\mathbb{C}}\backslash Z$, containing G_0. We claim that

$$0 < p(e) < 2\pi \quad \text{for every} \quad e \subset D. \tag{12}$$

The left inequality follows immediately from the definition of D. To prove the right inequality, we suppose without loss of generality that $e \subset \overline{\mathbb{U}}$, and use the tree S introduced in Section 2. Let $G \subset D$ be a face in $Q_{\mathbb{U}}$ whose boundary contains e. Then there is a path in S from the root q_0 to q_G. It is easy to see that all G_q for q in this path belong to D. The labels of all edges along this path are positive, because the whole path belongs to D. It follows from (5) that the labels of all edges of this path are less than 2π. Thus no edge in ∂G can have label 2π.

We put $B = \overline{\mathbb{C}}\backslash D$ and introduce an equivalence relation in $\overline{\mathbb{C}}$: $x \sim y$ if x and y belong to the same component of B. Let $Y = \overline{\mathbb{C}}/\sim$ be the factor space, and $w : \overline{\mathbb{C}} \to Y$ the projection map.

Since D is connected, every component of B is contractible, hence Y is a topological sphere, so we can identify it with the Riemann sphere. The symmetry $\mathbf{s} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an involution which leaves every point of $\mathbb{T}$ fixed. Since every component of B contains a vertex, it intersects $\mathbb{T}$. It follows that each component of B is symmetric. So Y also has an involution, such that w splits the involutions. This means that the identification of Y with $\overline{\mathbb{C}}$ can be made in such a way that

$$w : \overline{\mathbb{C}} \to Y \cong \overline{\mathbb{C}}, \quad w(x) = w(y) \quad \text{if and only if} \quad x \sim y \tag{13}$$

is symmetric. In particular $w(\mathbb{T}) = \mathbb{T}$. Furthermore, in view of (6), no component of B can contain two cubic roots of unity, so we can arrange that $w(v) = v$, for each cubic root v of 1. The cellular decomposition of $\overline{\mathbb{C}}$ defined by γ generates via w a cellular decomposition $X = X(p)$ of Y, so that the cells of X are $w(C)$, where C are the cells of the original decomposition. If the labeling p is nondegenerate, then w is a homeomorphism.

We are going to construct a continuous map $g^* : \overline{D} \to \mathbb{S} \cong \overline{\mathbb{C}}$, where $\mathbb{S}$ is another copy of the Riemann sphere. As a first step of our construction of g^*, we define a continuous map $\tilde{g} : \gamma \cap \overline{D} \to \mathbb{T} \subset \mathbb{S}$. To do this, we orient the edges of γ in the following way. Each edge $e \in E$ belongs to the boundaries of exactly two faces; let G be that one with $\sigma(G) = 1$. Then $e \subset \partial G$ by definition inherits positive orientation of ∂G.

We are going to define $\tilde{g}$, so that the following condition be satisfied for every $e \subset \overline{D}$:

$$\begin{aligned} &\textit{if } p(e) > 0, \textit{ then } \tilde{g} \textit{ maps } e \textit{ onto an arc of } \mathbb{T} \textit{ of length } p(e), \\ &\textit{homeomorphically, respecting orientation,} \\ &\textit{and if } p(e) = 0, \textit{ then } \tilde{g} \textit{ maps } e \textit{ into a point.} \end{aligned} \tag{14}$$

In particular the edges in ∂D are mapped into points, but closures of all edges in D are mapped homeomorphically onto their images. This follows from (12).

First we define $\tilde{g}$ on ∂G_0, so that condition (14) is satisfied, and $\tilde{g}(v_0) = 1$. Condition (5) with $G = G_0$ ensures that there is a unique way to define such continuous $\tilde{g}$ on ∂G_0. Furthermore, (6) implies that $\tilde{g}$ fixes all three cubic roots of 1.

Now we order all faces of γ in $D \cap \mathbb{U}$ into a sequence $(G_0, G_1, \ldots, G_m)$ so that for every $k = 1, \ldots, m$ the face G_k has exactly one common boundary edge e^* with

$$\bigcup_{j=0}^{k-1} \partial G_j. \tag{15}$$

The existence of such ordering was explained in Section 2, before (3).

Suppose that $\tilde{g}$ has been already defined on the edges in (15). In particular, it is defined on the edge $e^* \in \partial G_k$. Condition (5) with $G = G_k$ allows us to extend $\tilde{g}$ to all other edges in ∂G_k, so that (14) is satisfied.

After $\tilde{g}$ is defined for all edges in $\overline{\mathbb{U}}$, we extend it to the edges in $\overline{D} \backslash \overline{\mathbb{U}}$ by symmetry. This construction defines a symmetric continuous map $\tilde{g} : \gamma \cap \overline{D} \to \mathbb{T}$, which sends every component of ∂D to a point.

Notice that for every face G, the map $\tilde{g} : \partial G \to \mathbb{T}$ has degree ± 1, and is monotone; that is $\tilde{g}|_{\partial G}$ preserves or reverses the nonstrict cyclic order. As a next step, for each face $G \subset D$, we extend $\tilde{g}$ to a continuous map $g^* : \overline{G} \to \overline{\mathbb{U}} \subset \mathbb{S}$, if $\sigma(G) = 1$, or $g^* : \overline{G} \to \mathbb{S} \backslash \mathbb{U}$, if $\sigma(G) = -1$, so that the restriction on G is a homeomorphism onto the image. This can be done for every continuous monotone map $\partial G \to \mathbb{T}$ of degree ± 1.

It is clear, that this extension of $\tilde{g}$ into the interior of components $G \in Q$, $G \subset D$, can be made symmetrically; that is

$$g^* \circ \mathbf{s} = \mathbf{s} \circ g^*. \tag{16}$$

Finally we extend g^* to a continuous map $\overline{\mathbb{C}} \to \mathbb{S}$ so that it is constant on every component of the set $B = \overline{\mathbb{C}} \backslash D$. Then $g^*(x) = g^*(y)$ whenever $x \sim y$, the equivalence relation $\sim$ in (13). It follows that g^* factors as $g^* = g \circ w$, where w is the continuous map in (13). Here g is a continuous map $Y \to \mathbb{S}$.

If C is a cell of the cellular decomposition defined by γ, then w and g^* map C in the same way: either homeomorphically or to a point. It follows that g maps every closed cell of the form $w(\overline{C})$ homeomorphically onto the image. Furthermore, the cells $w(C)$ make a cellular decomposition X of Y, so g is a ramified covering. It can be ramified only at the vertices of X. If the labeling p is nondegenerate, that is w in (13) is a homeomorphism, all vertices of X have order 4, and g is ramified exactly at these vertices, having local degree 2 at each vertex.

There exists a unique conformal structure on Y, which makes g holomorphic. By the Uniformization theorem [1], [12], there exists a unique homeomorphism $\phi : Y \to \overline{\mathbb{C}}$, normalized by

$$\phi(e^{-2\pi i/3}) = e^{-2\pi i/3}, \quad \phi(1) = 1, \quad \phi(e^{2\pi i/3}) = e^{2\pi i/3}, \tag{17}$$

and such that $f = g \circ \phi^{-1}$ is a holomorphic map $\overline{\mathbb{C}} \to \mathbb{S}$, that is a rational function. It is easy to see that f is nonconstant and has degree at most d.

This function is the first component of $F_\gamma(p)$ in (10). The second component is

$$c : v \mapsto \phi \circ w(v), \quad v \in V, \tag{18}$$

which is a critical sequence in $\overline{\Sigma}_\gamma$. Indeed, by the symmetry property (16) and the symmetry of the normalization (17), ϕ is symmetric. Applying (16) again, we conclude that our rational function f is symmetric, and that all values of the function c belong to $\mathbb{T}$. An important consequence of our construction of F_γ is the following proposition, which we state in terms of function l as in (7):

Proposition 1. *Let $\Phi_\gamma : \overline{L}_\gamma \to \overline{\Sigma}_\gamma$ be the second component of the map F_γ, and $l = \Phi_\gamma(p)$ for some $p \in \overline{L}_\gamma$. Then $l(e) \neq 0$ if and only if $e \subset D \cap \mathbb{T}$.*

This follows from (18), taking into account that ϕ is a homeomorphism, and w collapses exactly those edges of γ which do not belong to D.

If the labeling p is nondegenerate, then the map w in (13), (18) is a homeomorphism, which implies that all $c(v)$, $v \in V$, are distinct and coincide with critical points of f. In this case we have $l(e) > 0$ for all edges $e \in E_{\mathbb{T}}$. So the second component Φ_γ of F_γ maps the set of nondegenerate labelings L_γ to the set of nondegenerate critical sequences Σ_γ, and γ is equivalent to $f^{-1}(\mathbb{T})$ via $\phi \circ w$, and we have (11).

Now we show that our map F_γ in (10) is well defined, that is f and c are independent of the choice of a labeled net within its equivalence class, and also independent of the extensions of g into the interiors of the components G. This independence follows from

LEMMA 2. *Let X_i, $i = 0, 1$, be cell complexes, h' a bijection between their cells such that $h'(\partial C) = \partial h'(C)$, Y a topological space and $f_i : X_i \to Y$ two continuous maps, whose restrictions to every closed cell are homeomorphisms onto the image, and $f_1(C) = f_0(h'(C))$ for every cell C in X_1. Then there exists a homeomorphism h such that $f_1 = f_0 \circ h$.*

Proof. We define h on every cell C in X_1 as $f^{-1}_{0,h'(C)} \circ f_1|_C$, where $f^{-1}_{0,h'(C)}$ is the inverse of the restriction $f_0|_{h'(C)} : h'(C) \to f_0(h'(C))$. □

Applying Lemma 2 to two rational functions f_0 and f_1, constructed from equivalent labeled nets, we conclude that $f_0 = f_1 \circ h$, where h is a homeomorphism of the Riemann sphere. This homeomorphism is evidently conformal and fixes three points; thus $h = \mathrm{id}$ and $f_1 = f_0$.

LEMMA 3. *The critical sequence $c = \Phi_\gamma(p)$ is well defined, that is it depends only on the class of labeled nets $([\gamma], p)$.*

Proof. Consider the cellular decomposition X, introduced after equation (13). If v is a vertex of X of degree at least 4, then $z = \phi(v)$ is a critical point of f, so $c(v)$ is well defined. Suppose now that $v^1, \ldots, v^m$ is a maximal chain of vertices of X of degree 2, which means that there are edges in X between these vertices, but no other edges connecting v^1 or v^m to vertices of degree 2. There is a unique way to extend this chain by adding v^0 and v^{m+1}, vertices of degree at least 4, so that v^0 is connected to v_1 and v^m to v^{m+1} by edges of X. Then $z^j = \phi(v^j)$, $j \in \{0, m+1\}$, are critical points of f, and $a_j = f(z^j)$, $j \in \{0, m+1\}$, corresponding critical values. The restriction of f onto the arc $[z^0, z^{m+1}] \subset \mathbb{T}$ maps this arc homeomorphically onto the arc $[a_0, a_{m+1}] \subset \mathbb{T}$. Then the position of the points $z^j = \phi(v_j)$, $j = 1, \ldots, m$, is determined from the fact that the length of each arc $[f(z^k), f(z^{k+1})] \subset \mathbb{T}$ is equal to $p(w^{-1}([v^k, v^{k+1}]))$, the label of an edge of γ. □

4. Continuity of Φ

For a fixed γ, the second component of our map F_γ in (10) is a map between two closed polytopes

$$\Phi : \overline{L} \to \overline{\Sigma}, \tag{19}$$

where $L = L_\gamma$, $\Sigma = \Sigma_\gamma$ and $\Phi = \Phi_\gamma$. In this section we prove that Φ is continuous.

Suppose that $p_1 \in \overline{L}$; we are going to prove that Φ is continuous at p_1. Let p_0 be a point close to p_1. Using the notation, similar to that introduced in Section 3, before (12), we consider the sets Z_i and the regions D_i, $i = 0, 1$.

In addition, let $B_1, \dots, B_m$ be the complete list of components of ∂D_1. Then $B_j \subset Z_1$ for $j = 1, \dots, m$. If p_0 is close enough to p_1, we may assume

$$Z_0 \subset Z_1, \quad \text{and thus} \quad D_1 \subset D_0. \tag{20}$$

We have the maps $w_i : \overline{\mathbb{C}} \to Y_i \cong \overline{\mathbb{C}},\ i = 0, 1$ as in (13), and $f_i = g_i \circ \phi_i^{-1}$, $i = 0, 1$, defined in Section 4. All maps involved in our argument are shown on the diagram below, where we use double notation $\mathbb{S} = \overline{\mathbb{C}}$ as in Section 3. For every vertex $v \in V$ the "critical value" $a(v) = g_1 \circ w_1(v)$ is defined,

$$a_k = a(v_k) = f_1 \circ \phi_1 \circ w_1(v_k) \in \mathbb{S},\ k = 1, \dots, 2d-2. \tag{21}$$

The actual set of critical values of f_1 is a subset of $\{a_k\}$ which might be proper.

$$\begin{tikzcd}
 & \overline{\mathbb{C}} \arrow[dl, "w_0"'] \arrow[dr, "w_1"] & \\
Y_0 \arrow[r, "g_0"] \arrow[d, "\phi_0"'] & \mathbb{S} & Y_1 \arrow[l, "g_1"'] \arrow[d, "\phi_1"] \\
\overline{\mathbb{C}} \arrow[ur, "f_0"] & & \overline{\mathbb{C}} \arrow[ul, "f_1"'] \arrow[ll, "h"']
\end{tikzcd}$$

We choose arbitrary $\delta > 0$. Then there exists $\varepsilon > 0$, such that the open discs U_k of radii ε around a_k are either disjoint or coincide, and have the property that every component K of the preimage of their union under f_1 has diameter less than δ. We set $\varepsilon_1 = \varepsilon/(8d)$ and suppose that

$$|p_1(e) - p_0(e)| < \varepsilon_1 \quad \text{for every} \quad e \in E. \tag{22}$$

In particular, in view of (20),

$$|p_0(e)| < \varepsilon_1 \quad \text{for} \quad e \subset \bigcup_{j=1}^{m} B_j. \tag{23}$$

The set

$$H = \mathbb{S} \backslash \bigcup_{k=1}^{2d-2} U_k \tag{24}$$

has a cell decomposition with two 2-dimensional cells C and C^*, where

$$\overline{C} = \overline{\mathbb{U}} \backslash \bigcup_{k=1}^{2d-2} U_k,$$

and C^* the symmetric cell to C. We choose 1-dimensional cells of this decomposition to be arcs of the unit circle and arcs of the circles ∂U_k, and for 0-dimensional cells we take the points of intersections of the circles ∂U_k with the unit circle.

Let H_1 be the preimage of the set H in (24) under f_1. Then H_1 has a cell decomposition, which is the preimage of our cell decomposition of (24), and $f_1|_{H_1}$ maps every cell of this decomposition homeomorphically. In fact $f_1|_{H_1}$ is a covering, because f_1 has no critical points in H_1.

It follows from (23) that

$$\text{diam}\, f_0 \circ \phi_0 \circ w_0(B_j) < \varepsilon/2, \quad \text{for each component } B_j \text{ of } \partial D_1, \tag{25}$$

because B_j is made of at most $4d-4$ edges e, whose labels $p_0(e)$ are at most ε_1 each. Furthermore, (22) and (21) imply that

$$\text{dist}\,(f_0 \circ \phi_0 \circ w_0(v_k), a_k) < \varepsilon/2, \quad k = 1, \ldots, 2d-2, \tag{26}$$

so f_0 has no critical values in H. As every B_j contains at least one vertex v_k of γ, we conclude from (25) and (26) that

$$f_0 \circ \phi_0 \circ w_0(B_j) \cap H = \emptyset, \quad j = 1, \ldots, m, \tag{27}$$

where H was defined in (24). Let H_0 be the component of $f_0^{-1}(H)$ intersecting $\phi_0 \circ w_0(G_0)$. As f_0 has no critical values in H, the restriction $f_0|_{H_0} : H_0 \to H$ is a covering. It follows from (27) that $H_0 \subset \phi_0 \circ w_0(D_1)$, because every component B_k of ∂D_1 is mapped by $f_0 \circ \phi_0 \circ w_0$ into $\overline{\mathbb{C}} \backslash H$.

The cell decomposition of H defined above pulls back to H_0, and f_0 maps each closed cell of this pullback onto a cell in H homeomorphically. Notice that each open cell of H_0 is contained in a unique cell of the form $\phi_0 \circ w_0(C)$ for some cell $C \subset D_1$ of γ. A similar statement holds for cells of H_1. This defines a bijection between cells of H_1 and those of H_0 which commutes with the boundary operator ∂. So Lemma 2 can be applied to $f_i|_{H_i}$. We conclude that

$$f_1 = f_0 \circ h \quad \text{on} \quad H_1, \tag{28}$$

where $h : H_1 \to H_0$ is a homeomorphism. Evidently h is holomorphic, and its boundary values on ∂H_1 belong to ∂H_0. Moreover, the components of ∂H_1 separating H_1 from the cubic roots of 1, are mapped to components of ∂H_0 separating the same cubic roots of 1 from H_0. Now we use the following

Lemma 4. *Suppose that a finite set $X = \{x_1, x_2, \ldots, x_n\} \subset \overline{\mathbb{C}}$, is given, such that $n \geq 3$, and x_1, x_2 and x_3 are the cubic roots of* 1. *Then for every $\eta > 0$ there exists $\delta \in (0, \eta)$ with the following property. Let $J_1, \ldots, J_n$ be disjoint open Jordan regions of diameter less than δ, $x_k \in J_k$, $k = 1, \ldots, n$, and h be an injective holomorphic function*

$$h : \overline{\mathbb{C}} \backslash \bigcup_{k=1}^{n} J_k \to \overline{\mathbb{C}},$$

such that for $k \leq 3$ the curves $h(\partial J_k)$ separate x_k from the two other cubic roots of 1. *Then*

$$\operatorname{dist}(h(z), z) < \eta, \quad \textit{whenever} \quad \operatorname{dist}(z, X) \geq \eta.$$

Proof (compare [4, Theorem 13]). Our proof is by contradiction. Suppose that there is a sequence $(\delta_j) \to 0$ and a sequence (h_j), which satisfies all conditions, but

$$\operatorname{dist}(h_j(z_j), z_j) \geq \eta \tag{29}$$

for some $\eta > 0$ and some points z_j with $\operatorname{dist}(z_j, X) \geq \eta$. It is easy to see that the closed domains $R_j = \overline{\mathbb{C}} \backslash \cup_{k=1}^{n} J_{k,j}$ tend to $\overline{\mathbb{C}} \backslash \{x_1, \ldots, x_n\}$ and that all functions h_j omit three cubic roots of 1 in their domains. By Montel's criterion [16], [12], (h_j) is a normal family and we can select a convergent subsequence. The limit h of this subsequence is a holomorphic injective function h in $\overline{\mathbb{C}} \backslash \{x_1, \ldots, x_n\}$, which omits the three cubic roots of 1. By the Great Theorem of Picard all points x_k are removable singularities, so h extends to a fractional-linear map. But this fractional-linear map also fixes three points, the cubic roots of 1, so it is the identity. This contradicts (29). □

Applying Lemma 4 to $h : H_1 \to \overline{\mathbb{C}}$, we obtain that $\operatorname{dist}(h(z), z) < \eta,\ z \in H_1$, so the critical sequences $(\phi_1 \circ w_1(v)),\ v \in V$ and $(\phi_0 \circ w_0(v)),\ v \in V$ are η-close. So our map (19) is continuous.

5. Boundary behavior of Φ

Our goal is to prove that $\Phi_\gamma : L_\gamma \to \Sigma_\gamma$ is surjective. This will be achieved with the help of Lemma 9 in Section 6. To verify that the conditions of this lemma are satisfied, we need to show that the preimages of closed faces of $\overline{\Sigma}_\gamma$ are homologically trivial. These preimages can be complicated, so we begin with an analysis of preimages of open faces. In this section, a net γ is fixed, so we do not show explicitly the dependence of various objects on γ.

Suppose that a convex polytope K is described by

$$\mathbf{A}x = b, \quad x \geq 0, \quad x \in \mathbb{R}^n,$$

where $\mathbf{A}$ is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. An *open face* of K is defined as

$$A_W = \{x \in K : x_j > 0,\ j \in W \text{ and } x_j = 0,\ j \notin W\}, \tag{30}$$

where W is a subset of $\{1, \ldots, n\}$, such that $A_W \neq \emptyset$. A *closed face* is the closure of an open face, and

$$\overline{A}_W = \{x \in K : x_j = 0,\ j \notin W\}. \tag{31}$$

Vertices are open faces and closed faces simultaneously. All open and closed faces are nonempty convex sets. We also notice that each closed face is a finite union of open faces.

We are going to apply these definitions to the convex polytope $\overline{\Sigma}$, described by (7), in the space of real-valued functions l on $E_{\mathbb{T}}$. First we state precisely which subsets $W \subset E_{\mathbb{T}}$ define open faces, that is for which W the set (30) is nonempty. It follows from (7) that the necessary and sufficient conditions are: $[v_0, v_1] \in W$, and that each of the two sequences

$$[v_1, v_2], \dots, [v_{N-1}, v_N] \quad \text{and} \quad [v_N, v_{N+1}], \dots, [v_{2d-3}, v_{2d-2}]$$

contains at least one edge in W. Using the notation e', e'' for the distinguished edges, introduced in Section 2, these conditions can be restated as

(a) $e' \in W$, and

(b) there is at least one edge in $W \backslash \{e'\}$ on each side of e''.

For a set $W \subset E_{\mathbb{T}}$, satisfying these conditions, we define the open face A_W of $\overline{\Sigma}$ by

$$A_W = \{l : l(e) > 0 \text{ for } e \in W \text{ and } l(e) = 0 \text{ for } e \in E_{\mathbb{T}} \backslash W\}. \tag{32}$$

We introduce a partial order on the set of open faces: $A_1 \prec A_2$ if and only if $\overline{A}_1 \subset \overline{A}_2$. From our definition (32) it follows that

$$A_{W_1} \prec A_{W_2} \quad \text{if and only if} \quad W_1 \subset W_2. \tag{33}$$

To characterize the preimage $\Phi^{-1}(A_W)$, of an open face, we use Proposition 1 from Section 3 and Lemma 5 below. To state this lemma, we need the following notation, similar to that used in Sections 3 and 4. For $p \in \overline{L}$, we define $Z(p)$ as the union of the closed edges e in E such that $p(e) = 0$, and $D(p)$ as the connected component of $\overline{\mathbb{C}} \backslash Z(p)$ containing G_0. Notice that $D(p)$ always contains at least three boundary edges of G_0, including e_{-1} and e_1. This follows from (5) with $G = G_0$ and (6). Let

$$E^0(p) = \{e \in E : e \subset \partial D(p) \cap \overline{\mathbb{U}}\} \tag{34}$$

and

$$E(p) = \{e \in E : e \subset D(p) \cap \mathbb{T}\}. \tag{35}$$

Figure 2 shows the part in $\overline{\mathbb{U}}$ of a net γ with $d = 5$, the set $E^0(p)$ (bold lines), and the set $E(p)$, which consists of the edges $[v_0, v_1], [v_2, v_3], [v_5, v_6]$, $[v_6, v_7]$ and $[v_7, v_8]$ on $\mathbb{T}$.

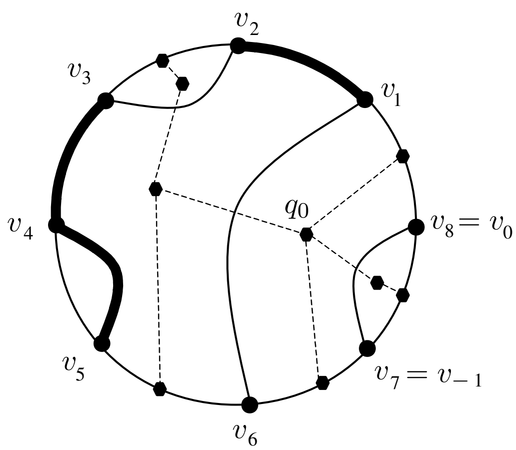

Figure 2: $S(p)$ in dotted lines, $E^0(p)$ in bold.

It is clear from the definitions (34) and (35), that the set $E^0(p)$ determines $E(p)$ uniquely. The opposite is also true:

LEMMA 5. *For* $p \in \overline{L}$, *the set* $E(p) \subset E$, *uniquely determines the set* $E^0(p)$.

Proof. We use the rooted tree $\hat{S}$, introduced in Section 2. Let $S(p)$ be the subtree of $\hat{S}$ spanned by q_0 and $\{q_e : e \in E(p)\}$. Figure 2 shows the tree $S(p)$ in dotted lines.

We claim that $S(p) = S'(p)$, where $S'(p)$ is the subtree of $\hat{S}$ spanned by $\{q_x : x \subset D(p) \cap \overline{\mathbb{U}}\}$ (here x may stand for a face or an edge of γ).

By definition of $E(p)$, we have $S(p) \subset S'(p)$. It remains to prove $S'(p) \subset S(p)$, which means that $D(p) \cap \overline{\mathbb{U}}$ contains exactly those faces $G \in Q_{\mathbb{U}}$ which have the property $q_G \in S(p)$. Thus $E(p)$ uniquely determines $D(p)$, and $D(p)$ uniquely determines $E^0(p)$.

To prove our claim, suppose that $S'(p) \not\subset S(p)$. Since both $S(p)$ and $S'(p)$ belong to the tree $\hat{S}$, there exists a leaf q of $S'(p)$ which does not belong to $S(p)$. If $q = q_e$, where $e \in E_{\mathbb{T}}$, then $e \subset D(p)$, hence q_e is a vertex of $S(p)$, in contradiction to our choice of q. Suppose now that $q = q_G$, where $G \subset D(p)$, is a face in $Q_{\mathbb{U}}$. Let S_q be the path in S connecting q and q_0. Conditions (5) imply that $0 < p(e) < 2\pi$ for every $e = e_\tau$, $\tau \in S_q$ (See also (12).) This implies that there is an edge $e \subset \partial G$ such that $\tau_e \notin S_q$ and $0 < p(e) < 2\pi$. Since $q \notin S(p)$, we have $\tau_e \notin S(p)$. If $e \subset \mathbb{T}$, we have a contradiction with the definition of $S(p)$. Otherwise, the other face in $Q_{\mathbb{U}}$, having the edge e on its boundary, belongs to $D(p)$, and G is not a leaf of $S'(p)$, again a contradiction. $\square$

It follows from Proposition 1 of Section 3 and (35) that the preimage of an open face $A_W \subset \overline{\Sigma}$ is

$$\Phi^{-1}(A_W) = \{p \in \overline{L} : E(p) = W\}. \tag{36}$$

By Lemma 5, for every $p \in \overline{L}$, the set $W = E(p)$ uniquely determines a set $E^0_W = E^0(p)$. According to the remark before Lemma 5, $E^0(p)$ uniquely determines $E(p)$. Thus (36) can be rewritten as

(37)

$$\Phi^{-1}(A_W) = \{p \in \overline{L} : p(e) = 0 \text{ for all } e \in E^0_W, \text{ and } p(e) > 0 \text{ for all } e \in W\}.$$

Now we prove that these preimages (37) are nonempty.

Lemma 6. *For each subset $W \subset E_{\mathbb{T}}$ satisfying* (a) *and* (b) *in the beginning of this section, there exists $p \in \overline{L}_\gamma$ such that $W = E(p)$.*

Proof. Given a subset W of edges of $\gamma \cap \mathbb{T}$ satisfying (a) and (b), let us define a subtree S_W of the tree $\hat{S}$, as the union of all paths in $\hat{S}$ connecting vertices q_e, for $e \in W$, with q_0. The labeling p is defined inductively along the tree $\hat{S}$, starting from the vertex q_0. As W contains at least one edge, other than e', at each side of e'', we have $\tau_{e''} \in S_W$, and there is at least one edge e of G_0, other than e' and e'', such that $\tau_e \in S_W$. Let $m \geq 1$ be the number of all such edges. We define $p(e) = 2\pi/(3m)$ for each of them, $p(e') = p(e'') = 2\pi/3$, and $p(e) = 0$ for all other edges of G_0. This guarantees that (5) is satisfied for G_0. Notice that $0 < p(e) < 2\pi$ for an edge $e \subset \partial G_0$ if and only if $\tau_e \in S_W$.

Suppose now that the values of $p(e)$ are defined for all edges of faces $G_q \in Q_{\mathbb{U}}$, with q in a subtree S' of S containing q_0, so that $0 < p(e) < 2\pi$ if and only if τ_e belongs to S_W, and (5) is satisfied. If $S' = S$, then the labeling p is complete. Otherwise, there exists a vertex q^* in $S \setminus S'$ which is an extremity of an edge τ^* of S with another extremity of τ^* being in S'. Let $G^* = G_{q^*}$ and $e^* = e_{\tau^*}$. Since an extremity of τ^* belongs to S', the label $p(e^*)$ is already defined.

If $p(e^*) = 2\pi$ or $p(e^*) = 0$, then τ^* does not belong to S_W; hence all other boundary edges of G^* do not belong to S_W. In the first case, we define $p(e) = 0$ for all edges $e \neq e^*$ of G^*. In the second case, we choose an edge $e^{**} \neq e^*$ of G^* and define $p(e^{**}) = 2\pi$ and $p(e) = 0$ for all other edges of G^*. Then (5) is satisfied for $G = G^*$.

If $0 < p(e^*) < 2\pi$, then τ^* belongs to S_W. Since $e^* \notin \mathbb{T}$, there is at least one other edge e of G^* such that τ_e belongs to S_W. Let $n \geq 1$ be the number of all such edges. We define $p(e) = (2\pi - p(e^*))/n$ for all these edges, and $p(e) = 0$ for all other edges $e \neq e^*$ of G^*. Again we have (5) for $G = G^*$.

Now the values of $p(e)$ are defined for all edges of faces $G_q \in Q_{\mathbb{U}}$, for the vertices q of a connected subtree S'' of S obtained by adding τ^* and q^* to S', which concludes our inductive step. We extend our labeling p to edges in $\overline{\mathbb{C}}\backslash\mathbb{U}$ by symmetry, so that (4) is satisfied. The labeling p constructed in this way satisfies (5), (6) and $W = E(p)$. □

The closure of the set (37) is

$$\overline{\Phi^{-1}(A_W)} = \{p \in \overline{L} : p(e) = 0 \text{ for all } e \in E^0_W\}, \tag{38}$$

which is nonempty and convex. Actually $\overline{\Phi^{-1}(A_W)}$ is a closed face of $\overline{L}$.

Now we begin a study the intersection pattern of these sets (38), which will be continued in Lemma 8.

LEMMA 7. *If $A_1 \succ A_2 \succ \ldots \succ A_k$ is a decreasing chain of open faces of $\overline{\Sigma}$, then the intersection of closures of their preimages $\overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_k)}$ is a nonempty convex subset of $\overline{L}$ (a closed face).*

Proof. The sets $\overline{\Phi^{-1}(A_j)}$ are convex, so their intersection is convex. It remains to verify that the intersection is nonempty.

We have $A_j = A_{W_j}$ for some $W_j \subset E_{\mathbb{T}}$. As in the proof of Lemma 6, for each $j \in [1, k]$, we define a subtree $S_j \subset \hat{S}$ as the union of all paths in $\hat{S}$ connecting vertices q_e, $e \in W_j$, with the root q_0. We also define E^0_j as the set of all edges e of γ, such that $\tau_e \in \hat{S}\backslash S_j$, and τ_e has a vertex in S_j. It is easy to check that these definitions are consistent with notation of Lemmas 5 and 6: if $W_j = E(p)$ then $E^0_j = E^0(p)$.

We have the following inclusions:

$$W_1 \supset \ldots \supset W_k, \quad \text{and} \quad S_1 \supset \ldots \supset S_k.$$

The first inclusion follows from the assumption of the Lemma and (33), the second follows from the first one, and the definition of S_j. We assume without loss of generality that $A_1 = \Sigma$, so $W_1 = E_{\mathbb{T}}$, and $S_1 = \hat{S}$. According to (38),

$$\overline{\Phi^{-1}(A_j)} = \{p : p(e) = 0 \text{ for all } e \in E^0_j\}.$$

Thus we have to show that there exists a labeling p, such that

$$p(e) = 0 \quad \text{for all} \quad e \in \bigcup_{j=1}^{k} E^0_j.$$

To construct this labeling p, we order the set of faces in $Q_{\mathbb{U}}$ into a sequence $G_0, \ldots, G_{d-1}$, such that for every $n \in [1, d-1]$, the face G_n has exactly one common boundary edge with the union of faces $G_0, \ldots, G_{n-1}$. (Such ordering was explained in Section 2, before (3).)

First we construct $p(e)$ for the edges e in ∂G_0, as it is done in the proof of Lemma 6, using W_k as W. For these edges e, we have $p(e) > 0$ if and only if $\tau_e \in S_k$. Hence $p(e) > 0$ implies $\tau_e \in S_j$, and thus $e \notin E_j^0$ for all $j \in [1, k]$.

Suppose that p is already defined for all boundary edges of faces $G_0, \ldots,$ G_{n-1}, for some $n < d$, so that

$$p(e) > 0 \quad \text{only if} \quad e \notin E_j^0, \quad \text{for all} \quad j \in [1, k]. \tag{39}$$

We want to extend p to the boundary edges of $G = G_n$, so that the property (39) is preserved.

Let $m \in [1, k]$ be the integer, such that $q_G \in S_m \backslash S_{m+1}$ ($S_{k+1} := \emptyset$). Consider the path Γ in the tree S from q_0 to q_G. Let $q_{G'}$ be the vertex on this path preceding q_G. Then $G' < G$ in the sense of the partial order defined in (2), and this implies by (3) that $G' \in \{G_0, \ldots, G_{n-1}\}$. There exists exactly one boundary edge e^* in $\partial G \cap \partial G'$. This is the only edge in ∂G, on which p is defined so far. Since $q_G \in S_m$, we have $q_{G'} \in \Gamma \subset S_m$. This implies $\tau_{e^*} \in S_m$. There is at least one more boundary edge e^{**} of G, such that $\tau_{e^{**}} \in S_m$. This is because all leaves of S_m are in $\mathbb{T}$; hence q_G is not a leaf. We define $p(e^{**}) = 2\pi - p(e^*)$, and $p(e) = 0$ for all edges on ∂G, other than e^* and e^{**}.

Notice that on this inductive step, the only new edge for which a positive value of p was defined is the edge e^{**}. Now we are going to prove that the condition (39) was preserved on the inductive step. Since $q_{e^{**}} \in S_m$, we have $q_{e^{**}} \in S_j$ and $e^{**} \notin E_j^0$ for $j \leq m$. Since $q_G \notin S_{m+1}$, e^{**} does not belong to any E_j^0 for $j > m$. Otherwise, the face $G'' \neq G$ such that $e^{**} \subset \partial G \cap \partial G''$ would have the property $q_{G''} \in S_j$; hence the path from $q_{G''}$ to q_0, which contains q_G, would belong to S_j, which is impossible since $q_G \notin S_j$.

This inductive procedure defines $p(e)$ for all edges. The labeling p we defined satisfies (39), as required. □

6. Surjectivity of Φ and proof of Theorem 2

In this section we use homology groups with integral coefficients. We call a topological space *homologically trivial* if it has the same homology groups as one point. In particular, such set is nonempty and connected. Nonempty convex sets are homologically trivial. Thus Lemma 7 of the previous section shows that our map Φ satisfies the conditions of the following

LEMMA 8. *Let $\Phi : \overline{L} \to \overline{\Sigma}$ be a continuous map of closed polytopes, such that for every $k \geq 1$ and for every decreasing chain $A_1 \succ \ldots \succ A_k$ of open faces of $\overline{\Sigma}$, the set*

$$\overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_k)}$$

is homologically trivial. Then for every $k \geq 1$ and for every decreasing chain $A_1 \succ \ldots \succ A_k$ of open faces of $\overline{\Sigma}$, the set

$$\overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_{k-1})} \cap \Phi^{-1}(\overline{A_k})$$

is homologically trivial. In particular, $\Phi^{-1}(\overline{A})$ *is homologically trivial for every open face* A.

To prove this lemma we need the following result from [9], Corollaire de Théorème 5.2.4 (of Leray). Suppose that a compact topological space K has a finite covering by its closed subsets $\{K_j\}$, such that all intersections

$$K_{j_1} \cap \ldots \cap K_{j_m} \tag{40}$$

are either empty or homologically trivial. The *nerve* of such a covering is defined as the simplicial complex, whose vertices are K_j and a subset of vertices $\{K_{j_1} \ldots, K_{j_m}\}$ defines a simplex if and only if the intersection (40) is nonempty. *Then K has the same homology groups as the nerve.*

Another version of this result was proved by K. Borsuk [5]. Borsuk's theorem assumes the nonempty intersections to be absolute retracts, and concludes that K is of the same homotopy type as the nerve. We can use either of these two results, but we prefer the homology version.

Proof of Lemma 8. We use induction on $d = \dim A_k$. For $d = 0$, A_k is one point, so $\overline{A_k} = A_k$, and the assumption of the lemma contains its conclusion in this case.

Suppose now that a chain $A_1 \succ \ldots \succ A_k$ is given, $\dim A_k = d \geq 1$, and the conclusion of the lemma holds for all decreasing chains whose last term is of dimension at most $d-1$. Consider the set

$$X = \overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_{k-1})} \cap \Phi^{-1}(\overline{A_k}) = X_0 \cup \{X_B : B \prec A_k,\ \dim B \leq d-1\},$$

where

$$X_0 = \overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_k)},$$

and

$$X_B = \overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_{k-1})} \cap \Phi^{-1}(\overline{B}),$$

for all open faces $B \prec A_k$ of dimension at most $d-1$. Then for every collection $B_1, \ldots, B_q$ of such open faces we have

$$X_0 \cap X_{B_1} \cap \ldots \cap X_{B_q} = \overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_k)} \cap \Phi^{-1}(\overline{B}), \tag{41}$$

where $\overline{B} = \overline{B_1} \cap \ldots \cap \overline{B_q}$ is a face of dimension at most $d-1$. The set (41) is homologically trivial if and only if B is nonempty, by the assumption of induction. Similarly,

$$X_{B_1} \cap \ldots \cap X_{B_q} = \overline{\Phi^{-1}(A_1)} \cap \ldots \cap \overline{\Phi^{-1}(A_{k-1})} \cap \Phi^{-1}(\overline{B}), \tag{42}$$

where $\overline{B} = \overline{B_1} \cap \ldots \cap \overline{B_q}$ is homologically trivial if and only if B is nonempty.

This means that the nerve of the closed covering $X_0 \cup_B X_B$ of X coincides with the nerve of the covering of $\overline{A_k}$ by the closures of open faces $B \prec A_k$, including $\overline{A_k}$ itself. By the corollary of Leray's theorem stated above, X has the same homology groups as $\overline{A_k}$, but $\overline{A_k}$ is nonempty and convex, so X is homologically trivial. $\square$

From Lemma 8 we conclude that the preimages of *closed* faces of $\overline{\Sigma}$ under Φ are homologically trivial, in particular, they are nonempty and connected. To complete the proof of Therem 2 we use the following

LEMMA 9. *Let $\overline{L}$ be a topological space, $\overline{\Sigma}$ a finite cell complex, all closed cells of $\overline{\Sigma}$ are homeomorphic to closed balls, and $\Phi : \overline{L} \to \overline{\Sigma}$ a continuous mapping such that the preimage of each closed cell in $\overline{\Sigma}$ is homologically trivial. Then Φ is surjective.*

Proof. Let $C_k(\overline{L})$ be the space of k-chains of $\overline{L}$ with integer coefficients. Let C_* be the corresponding chain complex, with the natural differential $\partial : C_k(\overline{L}) \to C_{k-1}(\overline{L})$. Let $C_*(\overline{\Sigma})$ be the corresponding chain complex for $\overline{\Sigma}$.

For every closed k-cell A of $\overline{\Sigma}$, we are going to construct a chain $W_A \in C_k(\overline{L})$ so that $\Phi(W_A) \subset A$, $\Phi(\partial W_A) \subset \partial A$ and $\Phi_*^A[W_A] = [A]$. Here Φ_*^A is the mapping $H_k(\Phi^{-1}(A), \Phi^{-1}(\partial A)) \to H_k(A, \partial A)$ induced by Φ, $[W_A]$ is the class of W_A in $H_k(\Phi^{-1}(A), \Phi^{-1}(\partial A))$, and $[A]$ is the class of A in $H_k(A, \partial A) \cong \mathbb{Z}$.

We proceed inductively on $k = \dim A$. For $k = 0$, the preimage of the vertex A is nonempty, and we take a point in $\Phi^{-1}(A)$ as W_A. Suppose that the chains W_A are defined for all cells A of $\overline{\Sigma}$ with $\dim A < k$, so that $\Phi_*^A[W_A] = [A]$ and $\partial W_A = W_{\partial A}$. Here a chain W_C, for a chain $C = \sum m_\nu A_\nu$, is defined as $\sum m_\nu W_{A_\nu}$.

Let A be a cell in $C_k(\overline{\Sigma})$. Due to the induction hypothesis, $W_{\partial A}$ is a cycle in $C_{k-1}(\overline{L})$, and $\Phi_*^{\partial A}[W_{\partial A}] = [\partial A]$. Here $\Phi_*^{\partial A}$ is the mapping $H_{k-1}(\Phi^{-1}(\partial A)) \to H_{k-1}(\partial A)$ induced by Φ. As $\Phi^{-1}(A)$ is a homologically trivial subcomplex of $\overline{L}$, there exists a chain $W_A \in C_k(\overline{L})$ such that $W_A \subset \Phi^{-1}(A)$ and $\partial W_A = W_{\partial A}$. From the commutative diagram

$$\begin{array}{ccc} H_k(\Phi^{-1}(A), \Phi^{-1}(\partial A)) & \xrightarrow{\partial} & H_{k-1}(\Phi^{-1}(\partial A)) \\ {\scriptstyle \Phi_*^A}\Big\downarrow & & {\scriptstyle \Phi_*^{\partial A}}\Big\downarrow \\ H_k(A, \partial A) & \xrightarrow{\partial} & H_{k-1}(\partial A) \end{array}$$

we have $\partial \Phi_*^A[W_A] = \Phi_*^{\partial A} \partial [W_A] = [\partial A]$. As $H_k(A, \partial A) \cong \mathbb{Z}$ is generated by $[A]$ and $\partial[A] = [\partial A] \neq 0$, this implies $\Phi_*^A[W_A] = [A]$.

To complete the proof, we have to show that the mapping $\Phi : W_A \to A$ is surjective, for any cell A of $\overline{\Sigma}$. If this is not so, then there exists an internal point $a \in A$ not covered by $\Phi(W_A)$. Since $A \setminus a$ is contractible to ∂A, this contradicts the condition $\Phi_*^A([W_A]) = [A] \neq 0$. $\square$

Proof of Theorem 2. It remains to summarize what has been done. In Section 3, for each net γ we constructed a map (10), which transforms labelings into pairs (f, c), where f is a rational function of the class R^*, and c a critical sequence. Restricting this map to nondegenerate labelings we obtain rational functions of the class $R_\gamma \subset R^*$, whose critical points are given by the nondegenerate sequence c (see (11)). By Proposition 1 in Section 3, degenerate labelings produce degenerate critical sequences, that is $\Phi^{-1}(\Sigma) = L$ in Lemma 9. This lemma implies that all possible critical sets, consisting of $2d-2$ points, can be obtained in this way. So each γ produces a rational function of the class R_γ with prescribed critical points. We conclude by Lemma 1, that the number of rational functions in R^*, with $2d-2$ prescribed critical points is at least u_d. This proves Theorem 2 because, as we saw in the end of Section 2, different functions from R^* are nonequivalent. □

7. Proof of Theorem 1

In this section we derive Theorem 1 from Theorems A and 2. The vector space Poly_d of polynomials of degree at most d with complex coefficients is identified with $\mathbb{C}^{d+1}$. Every pair (r, q) of nonproportional polynomials spans a 2-dimensional subspace in Poly_d.

To parametrize the equivalence classes of rational functions of degree d, we consider the Grassmannian $G(2, d+1)$, which is the set of all 2-dimensional subspaces in Poly_d, and the locus $D_1 \subset G(2, d+1)$ of those pairs of polynomials (r, q), for which $\deg r/q < d$. Then D_1 is an algebraic subvariety of $G(2, d+1)$ of codimension 1. Two pairs (r_i, q_i), $i = 1, 2$, represent the same point in $G(2, d+1)\backslash D_1$ if and only if the rational functions r_1/q_1 and r_2/q_2 are equivalent. Thus classes of rational functions of degree d are parametrized by $G(2, d+1)\backslash D_1$.

The Wronski determinant of two nonproportional polynomials

$$W(r, q) = \begin{vmatrix} r & q \\ r' & q' \end{vmatrix}$$

is a nonzero polynomial of degree at most $2d-2$, whose zeros are finite critical points of $f = r/q$, counting multiplicities, and common zeros of r and q. The common zeros of r and q are multiple zeros of $W(r, q)$. If two pairs of polynomials define the same point in $G(2, d+1)$, then the Wronskians of these pairs differ by a constant multiple. The set of all nonzero polynomials of degree at most $2d-2$, modulo proportionality, is parametrized by $\mathbb{C}P^{2d-2}$. Thus we have a regular map $\widetilde{W} : G(2, d+1) \to \mathbb{C}P^{2d-2}$, defined by taking the proportionality class of the Wronski determinant.

We show that $\widetilde{W}$ is a finite map [13, p. 177]. This fact is known [6], [11], but we include a short proof. We normalize our Wronskians, so that the

coefficient of the monomial of the smallest degree equals 1. Notice that each monomial z^n, where $0 \le n \le 2d-2$, has only finitely many preimages under $\widetilde{W}$, namely the 2-subspaces, generated by pairs (z^k, z^m), where $k+m = n+1$ and $k \ne m$. If (r,q) represents a point in $G(2,d+1)$, we consider the one-parametric family of points represented by (r_λ, q_λ), $\lambda \in \mathbb{C}^*$, where $r_\lambda(z) = r(\lambda z)$ and $q_\lambda(z) = q(\lambda z)$. Putting $w_\lambda = \widetilde{W}(r_\lambda, q_\lambda)$, we obtain $W(r_\lambda, q_\lambda)(z) = \lambda W(r,q)(\lambda z)$, and after normalization $w_\lambda(z) = \lambda^{-n-1} W(r,q)(\lambda z)$, where $n \in [0, 2d-2]$ is the smallest degree of monomials in $W(r,q)$. So $\dim \widetilde{W}^{-1}(w_\lambda) = \dim \widetilde{W}^{-1}(w_1)$ for $\lambda \in \mathbb{C}^*$, and

$$w_0 := \lim_{\lambda \to 0} w_\lambda \quad \text{is} \quad w_0(z) = z^n.$$

As the dimension of preimage is an upper semi-continuous function of the point [13, p. 138], for regular mappings into compact spaces, that is

$$\limsup_{\lambda \to 0} \dim \widetilde{W}^{-1}(w_\lambda) \le \dim \widetilde{W}^{-1}(w_0) = 0,$$

we conclude that $\dim \widetilde{W}^{-1}(w_1) = 0$ for every $w_1 \in \mathbb{C}P^{2d-2}$, so the preimages are finite, and the map $\widetilde{W}$ is finite.

Let $D_2 \subset \mathbb{C}P^{2d-2}$ be the locus of polynomials with multiple roots, or having smaller degree than $2d-2$. Notice that $\widetilde{W}(D_1) \subset D_2$. According to Theorem A, for every point w in $\mathbb{C}P^{2d-2} \backslash D_2$ we have $|\widetilde{W}^{-1}(w)| \le u_d$. On the other hand, our Theorem 1 implies that for every point w in the open set $\mathbf{V}$ in $\mathbb{R}P^{2d-2}$ formed by polynomials with $2d-2$ distinct real zeros the cardinality of $\widetilde{W}^{-1}(w) \cap G_{\mathbb{R}}(2, d+1)$ is at least u_d. Here $G_{\mathbb{R}}(2, d+1)$ stands for the 'real part' of the Grassmannian, that is the collection of those 2-dimensional subspaces which can be generated by pairs of real polynomials. This means that

$$\widetilde{W}^{-1}(\mathbf{V}) \subset G_{\mathbb{R}}(2, d+1).$$

On the other hand, for finite maps we have $\widetilde{W}^{-1}(w) = \lim_{w' \to w} \widetilde{W}^{-1}(w')$, so $\widetilde{W}^{-1}(\overline{\mathbf{V}}) \subset G_{\mathbb{R}}(2, d+1)$, where $\overline{\mathbf{V}}$ is the subset of $\mathbb{R}P^{2d-2}$ formed by polynomials with all real zeros. □

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